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SO(10) Unification in Non-Commutative Geometry

A. H. Chamseddine[1] * and J. Fröhlich[2]

[1] Theoretische Physik, Universität Zürich, CH 8001 Zürich Switzerland
[2] Theoretische Physik, ETH, CH 8093 Zürich Switzerland

Abstract

We construct an $SO(10)$ grand unified theory in the formulation of non-commutative geometry. The geometry of space-time is that of a product of a continuos four dimensional manifold times a discrete set of points. The properties of the fermionic sector fix almost uniquely the Higgs structure. The simplest model corresponds to the case where the discrete set consists of three points and the Higgs fields are $\underline{16}_s \times \overline{\underline{16}}_s$ and $\underline{16}_s \times \underline{16}_s$. The requirement that the scalar potential for all the Higgs fields not vanish imposes strong restrictions on the vacuum expectation values of the Higgs fields and thus the fermion masses. We show that it is possible to remove these constraints by extending the number of discrete points to six and adding a singlet fermion and a $\underline{16}_s$ Higgs field. Both models are studied in detail.

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1. Introduction

Grand unified theories provide an attractive mechanism to unify the weak, strong and electromagnetic interactions and put order into the representations of quarks and leptons. At present, the simplest models are based on $SU(5)$ [1] and $SO(10)$ gauge theories [2]. The second class of models has the advantage of including all the fermions (plus a right handed neutrino) in one representation. This advantage does not translate itself into a more predictive theory, because there are many possibilities to break $SO(10)$ down to $SU(3)\times U(1)_{\rm em}$ requiring many different and often complicated Higgs representations [3]. What is clearly needed in grand unified theories is a principle to put order into the Higgs sector. During the last few years, much effort has been directed towards this problem by studying unified theories as low-energy limits of the heterotic string . Although this is an attractive strategy, it has proven to be a difficult one, due to the fact that one must search for good models among the very large number of string vacua. We shall follow, instead, a different strategy.

It has been shown by Connes [4-5] and Connes and Lott [6-7] that the ideas of non-commutative geometry can be applied to, among other things, model building in particle physics. In particular, the Dirac operator, defined on the one-particle Hilbert space of quarks and leptons, is used to construct the standard $SU(3)\times SU(2)\times U(1)$ model with the Higgs field unified with the gauge fields. The space-time used in this constrution is a product of a Euclidean four-dimensional manifold by a discrete two-point space. If, in coming years, an elementary Higgs field is observed experimentally, one can turn the argument around and view it as an indication that space-time has the product structure proposed by Connes. One expects that this beautiful and highly symmetric construction would yield some predictions, in particular constraints among the coupling constants and particle masses, and indeed it does under certain circumstances [8]. (See also [9] for an alternative realization of Connes program). However, such relations can only be taken seriously once quantization is understood, or if one can stabilize the radiative corrections by, for example, supersymmetrising the theory. In a recent paper [10] it has been shown that, by a simple modification of the construction of Connes, it is possible to obtain unified models such as the $SU(5)$ and left-right $SU(2)_{\rm L}\times SU(2)_{\rm R}\times U(1)_{\rm B-L}$ theories. Other models such as the flipped $SU(5)\times U(1)$ model are also within reach of these constructions. The interesting case of $SO(10)$ was not treated, because it was not clear how to proceed in view of the fact that a realistic $SO(10)$ model requires complicated Higgs representations.

Meanwhile, it has turned out that the solution is fairly simple, and the construction of a realistic $SO(10)$ model will be the main concern of this paper. All the tools that will be used here are explained in references [10], and a self contained summary can be found in section 2 of the second item in reference [10] (The results contained there will be freely used in this paper.)

The plan of this paper is as follows. In section 2, we construct the Dirac operator associated with an $SO(10)$ gauge theory and show that the simplest model corresponds to a discrete space of three points. In section 3, the symmetry breaking chain is described in detail and the vevs of the Higgs fields are given. In section 4, the potential is analyzed, and it is shown that a potential survives after eliminating the auxiliary fields only if the vevs of the Higgs fields satisfy certain constraints. In section 5, we show that it is possible to relax the constraints, provided that the number of discrete points is taken to be six and certain symmetries are imposed.

2. The SO(10) framework

The starting point in Connes' construction [4-8] is the specification of the fermionic sector and the Dirac operator on the space of spinors. In the $SO(10)$-model [2], the fermions neatly fit in the $\underline{16}_s$ spinor representation, repeated three times. A single fermionic family is described by the field $\psi_{\alpha\hat{\alpha}}$, where α is an $SO(1,3)$ Lorentz spinor index with four components and $\hat{\alpha}$ is an $SO(10)$ spinor index with thirty two components. It satisfies both space-time and $SO(10)$ chirality conditions:

$$\begin{aligned} (\gamma_5)_\alpha^\beta \psi_{\beta\hat{\alpha}} &= \psi_{\alpha\hat{\alpha}} \\ (\Gamma_{11})_{\hat{\alpha}}^{\hat{\beta}} \psi_{\alpha\hat{\beta}} &= \psi_{\alpha\hat{\alpha}}. \end{aligned} \tag{2.1}$$

where $\gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3$, $\Gamma_{11} = -i\Gamma_0\Gamma_1\cdots\Gamma_9$, and for later convenience we have denoted Γ_{10} by Γ_0. This reduces the independent spinor components to two for the space-time indices, and to sixteen for the $SO(10)$ indices. The general fermionic action is given by

$$\overline{\psi^p_{\alpha\hat{\alpha}}}(\partial\!\!\!/ + A^{IJ}\Gamma_{IJ})^{\beta\hat{\beta}}_{\alpha\hat{\alpha}}\psi^p_{\beta\hat{\beta}} \ + \psi^{Tp}_{\alpha\hat{\alpha}} C^{\alpha\beta} H^{pq}_{\hat{\alpha}\hat{\beta}} \psi^q_{\beta\hat{\beta}} \tag{2.2}$$

where C is the charge conjugation matrix, $p, q = 1, 2, 3$ are family indices, and H is some appropriate combination of Higgs fields breaking the subgroup $SU(2) \times U(1)$ of $SO(10)$ at low energies. An exception of a Higgs field that breaks the symmetry at high energies and yet couples to fermions is the one that gives a Majorana mass to the right handed neutrinos [11]. The other Higgs fields needed to break the $SO(10)$ symmetry at high energies should not couple to the fermions so as not to give the quarks and leptons super heavy masses.

From the form of eq.(2.2) we deduce that the gauge and Higgs fields are valued in the Clifford algebra of $SO(10)$, projected with the chirality operator acting on the right and the left of the fields. Since we know that in the non-commutative construction the Higgs

fields are obtained by having more than one copy of Minkowski space, we need to choose a discrete space containing at least three points. On two of the copies, the associated spinors are taken to be identical, and the Higgs fields in this direction will not couple to the fermions as these have the same chirality. On the third copy the fermions are taken to be the conjugate spinors, as can be deduced from the second term of eq (2.2). Thus, between copies one and two, we must impose a permutation symmetry, while between copies one and three we must require some form of conjugation symmetry. If we insist that the fermionic sector exhibit a Z_2-symmetry then four copies of Minkowski space are necessary, with the third and fourth copies identified, too. This option will be pursued in the last section. Since both $SO(1,3)$ and $SO(10)$ have conjugation matrices, we take the conjugate spinor to be given by

$$\psi^c \equiv BC\overline{\psi}^T \tag{2.3}$$

where B is the $SO(10)$ conjugation matrix satisfying $B^{-1}\Gamma_I B = -\Gamma_I^T$. Thus the spinor for the system is given by

$$\Psi = \begin{pmatrix} \psi \\ \psi \\ \psi^c \end{pmatrix} \tag{2.4}$$

The chirality conditions on the spinor Ψ are given by

$$\begin{aligned} \gamma_5 \otimes \mathrm{diag}(1,1,-1)\Psi &= \Psi \\ \gamma_5 \otimes \Gamma_{11}\Psi &= \Psi \end{aligned} \tag{2.5}$$

Before proceeding, it is useful to address the problem of neutrino masses. The right handed neutrino must acquire a large mass. This is usually done by coupling the fermions to a $\underline{126}$ or to a $\underline{16}_s$ Higgs field with appropriate vacuum expectation values (vev's) giving a mass to the right handed neutrino but not to the remaining fermions. The $\underline{126}$ appears already with the Higgs fields that give masses to the fermions. The $\underline{16}_s$ can only be obtained by extending the fermionic space with a singlet spinor. This implies that the number of copies of Minkowski space must be increased by one or two, depending on whether the Z_2 symmetry is required or not. In this case, two of the neutral fermions will become superheavy, while the third would remain massless.

The fermionic space is then chosen to be

$$\begin{pmatrix} \psi \\ \psi \\ \psi^c \\ \psi^c \\ \lambda \\ \lambda^c \end{pmatrix} \tag{2.6}$$

where the number of copies associated with conjugate spinors is doubled. We shall first consider a spinor space corresponding to eq. (2.4) and treat the more complicated case corresponding to eq. (2.6) in the last section.

We are now ready to specify a triple $(\mathcal{A}, h, D)$ defining a non-commutative geometry, where h is the Hilbert space of the spinors Ψ, $\mathcal{A}$ is an involutive algebra of operators on h, and D is an unbounded, self-adjoint operator on h [4-5]. Let X be a compact Riemannian four-dimensional spin-manifold, $\mathcal{A}_1$ the algebra of functions on X and (h_1, D_1, Γ_1) the Dirac K cycle, with $h_1 \equiv L^2(X, \sqrt{g}d^4x)$, on $\mathcal{A}_1$, and Γ_1 is a Z_2 grading. We choose $\mathcal{A}$ to be given by

$$\mathcal{A}_2 = P_+ \text{Cliff}\big(SO(10)\big) P_+ \tag{2.7}$$

where $P_\pm = \frac{1}{2}(1 \pm \Gamma_{11})$, and set

$$\mathcal{A} = \mathcal{A}_1 \otimes \mathcal{A}_2$$

We define $\Omega^*(\mathcal{A}) = \oplus_{n=0}^{\infty} \Omega^n(\mathcal{A})$ to be the universal differential algebra over $\mathcal{A}$, with $\Omega^0(\mathcal{A}) = \mathcal{A}$, and

$$\Omega^n(\mathcal{A}) = \{\sum_i a_0^i da_1^i \ldots da_n^i \; : a_j^i \epsilon \mathcal{A}, \forall i, j\}, \qquad n = 1, 2, \cdots.$$

Thus, an element $\rho \epsilon \Omega^1(\mathcal{A})$ has the form

$$\rho = \sum_i a^i db^i, \tag{2.8}$$

and we impose the condition

$$\sum_i a^i b^i = 1,$$

since $d1 = 0$. Let π_0 denote the representation of $\mathcal{A}$ on the space $h_1 \otimes h_2$ of square integrable spinors for $SO(1,3) \times SO(10)$, where h_2 is the 32-dimensional Hilbert space on which $\mathcal{A}_2$ acts. Let $\overline{\pi_0}$ denote the anti-representation given by

$$\overline{\pi_0}(a) = B\overline{\pi_0(a)}B^{-1}. \tag{2.9}$$

We then define $\pi(a)$ by

$$\pi(a) = \pi_0(a) + \pi_0(a) + \overline{\pi_0}(a) \tag{2.10}$$

acting on the Hilbert space

$$\tilde{h} = h_1 \otimes \big(h_2^{(1)} \oplus h_2^{(2)} \oplus h_2^{(3)}\big),$$

where $h_2^{(i)} \cong h_2, \qquad i = 1, 2, 3$. Let h denote the subspace of $\tilde{h}$ which is the image of the orthogonal projection onto elements of the form

$$\begin{pmatrix} P_+\psi \\ P_+\psi \\ P_-\psi^c \end{pmatrix}$$

in $\tilde{h}$. Clearly, h is invariant under $\pi(\mathcal{A})$. (One can think of h as being a space of sections of a "vector bundle" over $\mathcal{A}$.) On $\tilde{h}$ we define a self-adjoint Dirac operator D by setting

$$D = \begin{pmatrix} \partial\!\!\!/ \otimes 1 \otimes 1 & \gamma_5 \otimes M_{12} \otimes K_{12} & \gamma_5 \otimes M_{13} \otimes K_{13} \\ \gamma_5 \otimes M_{21} \otimes K_{21} & \partial\!\!\!/ \otimes 1 \otimes 1 & \gamma_5 \otimes M_{23} \\ \gamma_5 \otimes M_{31} \otimes K_{31} & \gamma_5 \otimes M_{32} \otimes K_{32} & \partial\!\!\!/ \otimes 1 \otimes 1 \end{pmatrix} \tag{2.11}$$

where the K_{mn} are 3×3 family-mixing matrices commuting with the $\pi(\mathcal{A})$. We impose the symmetries $M_{12} = M_{21} = \mathcal{M}_0$, $M_{13} = M_{23} = \mathcal{N}_0$, $M_{31} = M_{32} = \mathcal{N}_0^*$, with $\mathcal{M}_0 = \mathcal{M}_0^*$. Similar conditions are imposed on the matrices K_{mn}. For D to leave the subspace h invariant, $\mathcal{M}_0$ and $\mathcal{N}_0$ must have the form

$$\begin{aligned} \mathcal{M}_0 &= P_+\big(m_0 + im_0^{IJ}\Gamma_{IJ} + m_0^{IJKL}\Gamma_{IJKL}\big)P_+ \\ \mathcal{N}_0 &= P_+\big(n_0^I\Gamma_I + n_0^{IJK}\Gamma_{IJK} + n_0^{IJKLM}\Gamma_{IJKLM}\big)P_- \end{aligned} \tag{2.12}$$

where

$$\Gamma_{I_1 I_2 \cdots I_n} = \frac{1}{n!}\Gamma_{[I_1}\Gamma_{I_2}\cdots\Gamma_{I_n]}$$

are antisymmetrized products of the gamma matrices.

Next we define an involutive "representation" $\pi : \Omega^*(\mathcal{A}) \leftarrow B(h)$ of $\Omega^*(\mathcal{A})$ by bounded operators on h; ($B(h)$ is the algebra of bounded operators on h): We set

$$\pi_0(a_0 da_1 da_2 \cdots da_n) = \pi(a_0)[D, \pi(a_1)][D, \pi(a_2)] \cdots [D, \pi(a_n)]. \tag{2.9}$$

The image of a one-form ρ is

$$\pi(\rho) = \sum_i a^i[D, b^i], \qquad \sum_i a^i b^i = 1. \tag{2.14}$$

From now on, we shall write a^i and b^i, instead of $\pi(a^i)$ and $\pi(b^i)$, respectively. Every one-form ρ determines a connection, ∇, on h: We set

$$\nabla = D + \pi(\rho). \tag{2.15}$$

The curvature of ∇ is then given by

$$\theta = \pi(d\rho) + \pi(\rho^2) \tag{2.16}$$

where

$$\pi(d\rho) = \sum_i [D, \pi(a^i)][D, \pi(b^i)].$$

It is straightforward to compute $\pi(\rho)$ and one gets [10]

$$\pi(\rho) = \begin{pmatrix} A & \gamma_5 \mathcal{M} K_{12} & \gamma_5 \mathcal{N} K_{13} \\ \gamma_5 \mathcal{M} K_{12} & A & \gamma_5 \mathcal{N} K_{23} \\ \gamma_5 \mathcal{N}^* K_{31} & \gamma_5 \mathcal{N}^* K_{32} & B\overline{A}B^{-1} \end{pmatrix} \tag{2.17}$$

where the fields A, $\mathcal{M}$ and $\mathcal{N}$ are given in terms of the a^i and b^i by

$$\begin{aligned} A &= P_+ (\sum_i a^i \partial\!\!\!/ b^i) P_+ \\ \mathcal{M} + \mathcal{M}_0 &= P_+ (\sum_i a^i \mathcal{M}_0 b^i) P_+ \\ \mathcal{N} + \mathcal{N}_0 &= P_+ (\sum_i a^i \mathcal{N}_0 B \overline{b^i} B^{-1}) P_- \end{aligned} \tag{2.18}$$

We can expand these fields in terms of the $SO(10)$ Clifford algebra as follows:

$$\begin{aligned} A &= P_+ \big(ia + a^{IJ} \Gamma_{IJ} + i a^{IJKL} \Gamma_{IJKL} \big) P_+ \\ \mathcal{M} &= P_+ \big(m + i m^{IJ} \Gamma_{IJ} + m^{IJKL} \Gamma_{IJKL} \big) P_+ \\ \mathcal{N} &= P_+ \big(n^I \Gamma_I + n^{IJK} \Gamma_{IJK} + n^{IJKLM} \Gamma_{IJKLM} \big) P_- \end{aligned} \tag{2.19}$$

The self-adjointness condition on $\pi(\rho)$ implies, after using the hermiticity of the Γ_I matrices, that all the fields appearing in the expansion of $A, \mathcal{M}$ are real,because both are self-adjoint, while those in $\mathcal{N}$ are complex. The tracelessness condition on $\mathrm{tr}(\Gamma_1 \pi(\rho))$ where Γ_1 is the grading operator given in the first equation of (2.5). This restricts $a = 0$ and then this corresponds to the gauge theory of $SU(16)$. In this case one must also add mirror fermions to cancel the anomaly and will not be considered here. We shall require instead that the gauge fields acting on the first and third copies have identical components in the Clifford algebra basis. Since

$$B\overline{A}B^{-1} = P_- (-ia + a^{IJ} \Gamma_{IJ} - i a^{IJKL} \Gamma_{IJKL}) P_- \tag{2.20}$$

This implies that

$$\begin{aligned} a_\mu &= 0 \\ a_\mu^{IJKL} &= 0 \end{aligned} \tag{2.21}$$

The above rquirement can be understoood as the physical condition that the fermions in the first and third copies will have identical coupling to the gauge fields. Then the fermionic action will be given by

$$(\Psi, \mathcal{P}(d+\rho)\mathcal{P}\Psi) = \int d^4x \Psi^*(x)\mathcal{P}\big(D+\pi(\rho)\big)\mathcal{P}\Psi(x) \tag{2.22}$$

where

$$\mathcal{P} = \mathrm{diag}(P_+, P_+, P_-).$$

To transform this expression from Euclidean space to Minkowski space in order to impose the space-time chirality condition, we have to perform the following substitutions: $\gamma^4 \to i\gamma^0$, $\gamma_5 \to -i\gamma_5$, $\psi^* \to \overline{\psi}$, $\psi^{c*} \to -\overline{\psi^c}$. Because of space-time chirality, the field $\mathcal{M}$ decouples from the fermions. Then this is the field that must acquire a vacuum expectation value breaking $SO(10)$ at very large energies. The field $\mathcal{N}$ does couple to fermions and must acquire expectation values that gives the small fermionic masses, except for possible large values of the components that give a mass to the right-handed neutrino.

Now we are ready to write the fermionic action in terms of the component fields

$$\begin{aligned} I_{\mathrm{f}} = \int d^4x \Big(2\, \overline{\psi_+} \Big[i(\partial\!\!\!/ + A)\psi_+ + \gamma_5(\mathcal{N}+\mathcal{N}_0)\psi_+^c K_{13} \Big] \\ + \overline{\psi_+^c} \Big[i(\partial\!\!\!/ + A)\psi_+^c + \gamma_5(\mathcal{N}^*+\mathcal{N}_0^*)\psi_+ K_{13}^* \Big] \Big) \end{aligned} \qquad (2.23).$$

where $\psi_+ = P_+\psi$ and by $SO(10)$ chirality is equal to ψ. From here on and when convenient we shall denote $\mathcal{M}$ by $P_+\mathcal{M}P_+$ and $\mathcal{N}$ by $P_+\mathcal{N}P_-$. Equation (2.23) can be simplified by using the proporties of the charge conjugation matrices B and C:

$$\begin{aligned} B^{-1}\Gamma_I B &= -\Gamma_I^T \\ C^{-1}\gamma_\mu C &= -\gamma_\mu^T. \end{aligned} \tag{2.24}$$

After rescaling $\psi \to \frac{1}{\sqrt{3}}\psi$ the action (2.23) simplifies to

$$I_{\mathrm{f}} = \int d^4x \Big(\overline{\psi_+} i(\partial\!\!\!/ + A)\psi_+ - \frac{1}{\sqrt{3}} \big(\psi_+^T B^{-1} C^{-1} (\mathcal{N}^* + \mathcal{N}_0^*)\psi_+ K_{13}^* + h.c \big) \Big) \tag{2.25}$$

Thus we have achieved our goal of constructing a Dirac operator that gives the appropriate interactions of an $SO(10)$ unified gauge theory.

3. The SO(10) symmetry breaking

The symmetry breaking pattern that breaks the gauge group $SO(10)$ must be coded into the Dirac operator D. The Higgs fields at our disposal are $\mathcal{M}$, and $\mathcal{N}$. In terms of $SO(10)$ representations these are $\underline{1}$, $\underline{45}$, $\underline{210}$ in $\mathcal{M}$, and complex $\underline{10}$, $\underline{120}$ and $\underline{126}$ in $\mathcal{N}$. To be explicit we shall work in a specific Γ matrix representation first introduced by Georgi and Nanopolous [2]. The 32×32 Γ matrices are represented in terms of tensor products of five sets of Pauli matrices $\sigma_i, \tau_i, \eta_i, \rho_i, \kappa_i$ where $i = 1, 2, 3$. To these matrices we assign the following matrices on the tensor product space:

$$\begin{aligned}
\sigma_i &\to 1_2 \otimes 1_2 \otimes 1_2 \otimes 1_2 \otimes \sigma_i \\
\tau_i &\to 1_2 \otimes 1_2 \otimes 1_2 \otimes \tau_i \otimes 1_2 \\
\eta_i &\to 1_2 \otimes 1_2 \otimes \eta_i \otimes 1_2 \otimes 1_2 \\
\rho_i &\to 1_2 \otimes \rho_i \otimes 1_2 \otimes 1_2 \otimes 1_2 \\
\kappa_i &\to \kappa_i \otimes 1_2 \otimes 1_2 \otimes 1_2 \otimes 1_2
\end{aligned} \tag{3.1}$$

The Γ matrices are then given by

$$\begin{aligned}
\Gamma_i &= \kappa_1 \rho_3 \eta_i \\
\Gamma_{i+3} &= \kappa_1 \rho_1 \sigma_i \\
\Gamma_{i+6} &= \kappa_1 \rho_2 \tau_i \\
\Gamma_0 &= \kappa_2 \\
\Gamma_{11} &= \kappa_3
\end{aligned} \tag{3.2}$$

where $i = 1, 2, 3$, and when it is obvious we shall omit the tensor product symbols. In this basis an $SO(10)$ chiral spinor will take the form

$$\psi_+ = \begin{pmatrix} \chi_+ \\ 0 \end{pmatrix} \tag{3.3}$$

where χ is a $\underline{16}_s$ in the space $V_\rho \otimes V_\eta \otimes V_\tau \otimes V_\sigma$, with $V_\rho \equiv \cdots \equiv V_\sigma \equiv C^2$. The $SO(10)$ conjugation matrix is defined by $B \equiv -\Gamma_1 \Gamma_3 \Gamma_4 \Gamma_6 \Gamma_8$ which, in the basis of equation (3.2), becomes

$$B = \kappa_1 \rho_2 \eta_2 \tau_2 \sigma_2 \equiv \kappa_1 b \tag{3.4}$$

where the matrix $b = \rho_2 \eta_2 \tau_2 \sigma_2$ is the conjugation matrix in the space of the sixteen component spinors. The action of B on a chiral spinor is then

$$B\psi_+ = \begin{pmatrix} 0 \\ b\chi_+ \end{pmatrix} \tag{3.5}$$

The advantage of this system of matrices is that both spinors, χ_+ and $bC\overline{\chi_+}^T$, have the same form, except for the first one is left-handed and the second one is right-handed. To correctly associate the components of χ_+ with quarks and leptons, we consider the action of the charge operator [3] on χ_+:

$$\begin{aligned} Q &= \frac{i}{6}(\Gamma_{45} + \Gamma_{69} + \Gamma_{78}) - \frac{i}{2}\Gamma_{12} \\ &= -\frac{1}{6}(\sigma_3 + \tau_3 + \rho_3\tau_3\sigma_3) + \frac{1}{2}\eta_3 \end{aligned} \tag{3.6}$$

which gives

$$Q\chi_+ = \mathrm{diag}(0, \frac{2}{3}, \frac{2}{3}, \frac{2}{3}, -1, -\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 1, -\frac{2}{3}, -\frac{2}{3}, -\frac{2}{3}, 0)\chi_+ \tag{3.7}$$

Thus the components of the left handed spinor χ_+ are written as

$$\chi_+ = \begin{pmatrix} n_L \\ u_L^1 \\ u_L^2 \\ u_L^3 \\ e_L \\ d_L^1 \\ d_L^2 \\ d_L^3 \\ -(d_R^3)^c \\ (d_R^2)^c \\ (d_R^1)^c \\ -(e_R)^c \\ (u_R^3)^c \\ -(u_R^2)^c \\ -(u_R^1)^c \\ (n_R)^c \end{pmatrix} \tag{3.8}$$

where the c in this equation stands for the usual charge conjugation, eg. $d^c = C\overline{d}^T$. The upper and lower components in χ are mirrors, with the signs chosen so that the spinor $bC\overline{\chi_+}^T$ has exactly the same form as χ_+, but with the left-handed and right handed signs, L and R, interchanged.

We now specify the vacuum expectation values (vevs) $\mathcal{M}_0$ and $\mathcal{N}_0$. The group $SO(10)$ is broken at high energies by $\mathcal{M}$ which contains the representations $\underline{45}$ and $\underline{210}$. By taking the vev of the $\underline{210}$ to be $\mathcal{M}^{0123} = O(M_G)$, the $SO(10)$ symmetry is broken to $SO(4) \times SO(6)$ which is isomorphic to $SU(4)_c \times SU(2)_L \times SU(2)_R$. The $SU(4)_c$ is further broken to $SU(3)_c \times U(1)_c$ by the vev of the $\underline{45}$. Therefore we write [2-3]

$$\begin{aligned} P_+\mathcal{M}_0P_+ &= P_+\Big(M_G\Gamma_{0123} - iM_1(\Gamma_{45} + \Gamma_{78} + \Gamma_{69}\Big)P_+ \\ &= \frac{1}{2}(1 + \kappa_3)\Big(-M_G\rho_3 + M_1(\sigma_3 + \tau_3 + \rho_3\tau_3\sigma_3)\Big) \end{aligned} \tag{3.9}$$

Therefore $\mathcal{M}_0$ breaks $SO(10)$ to $SU(3)_c \times U(1)_c \times SU(2)_L \times SU(2)_R$ which is also of rank five. The rank is reduced by giving a vev to the components of $\underline{126}$ that couple to the right-handed neutrino.Therefore the vev of $\mathcal{N}_0$ must contain the term

$$M_2(\frac{1}{2^5})(\kappa_1 + i\kappa_2)(\rho_1 + i\rho_2)(\eta_1 + i\eta_2)(\tau_1 + i\tau_2)(\sigma_1 + i\sigma_2) \qquad (3.10)$$

In terms of the gamma matrices, equation (3.8) has a rather complicated form

$$\frac{1}{8}\Big(((\Gamma_{13489} + i(1 \to 2)) + i(4 \to 5)) - i(8 \to 7)\Big). \qquad (3.11)$$

The vev of $\mathcal{N}_0$ break $U(1)_c \times SU(2)_R$ to $U(1)_Y$, and the surviving group would be the familiar $SU(3)_c \times SU(2)_L \times U(1)_Y$. The generators of $SU(2)_L \times SU(2)_R$ are [2]

$$\begin{aligned} T^i_{L,R} &= -\frac{i}{2}(\frac{1}{2}\epsilon^{ijk}\Gamma_{jk} \pm \Gamma^{i0}) \\ &= \frac{1}{2}(1 \pm \kappa_3\rho_3)\eta^i, \end{aligned} \qquad (3.12)$$

while $SU(4)_c$ is generated by

$$\begin{aligned} -i\Gamma_{i+3,j+3} &= \epsilon_{ijk}\sigma^k \\ -i\Gamma_{i+6,j+6} &= \epsilon_{ijk}\tau^k \\ -i\Gamma_{i+3,j+6} &= \rho_3\tau_j\sigma_i. \end{aligned} \qquad (3.13)$$

It is straightforward to check that the only generators that leave $\mathcal{M}_0$ and the part of $\mathcal{N}_0$ given by (3.10) invariant are those of the standard model. We shall explicitly identify these generators, in order to proceed to the next stage of breaking $SU(2)_L \times U(1)_Y$, without any ambiguity. The eight $SU(3)$ generators are given by $(1-\rho_3\tau_3)\sigma_i$, $(1-\rho_3\sigma_3)\tau_i$, $\rho_3(\tau_1\sigma_1 + \tau_2\sigma_2)$ and $\rho_3(\tau_2\sigma_1 - \tau_1\sigma_2)$. Finally the $U(1)_Y$ generator is

$$Y = -\frac{1}{3}(\sigma_3 + \tau_3 + \rho_3\tau_3\sigma_3) + \frac{1}{2}(1 - \kappa_3\rho_3)\eta_3, \qquad (3.14)$$

and its action on the spinor χ_+ is given by

$$Y\chi_+ = \mathrm{diag}(-1, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, -1, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{2}{3}, \frac{2}{3}, \frac{2}{3}, 2, -\frac{4}{3}, -\frac{4}{3}, -\frac{4}{3}, 0)\chi_+ \qquad (3.15)$$

This is related to the charge operator Q by

$$Q = \frac{1}{2}Y + T^3_L \qquad (3.16)$$

where the action of the $SU(2)_L$ isospin T^3_L on χ_+ is given by $T^3_L = \frac{1}{2}(1 + \rho_3)\eta_3$.

For the last stage of symmetry breaking of $SU(2)_L \times U(1)_Y$ we can use the field $\mathcal{N}$ which contains the compex representations $\underline{10}$, $\underline{120}$ and $\underline{126}$. The most general vev that preserves the group $SU(3)_c \times U(1)_Q$ is

$$\begin{aligned} P_+\mathcal{N}_0P_- = &\frac{1}{2}(1+\kappa_3)\Big(\big(is\Gamma_0 + p\Gamma_3\big) \\ &+ \big(a'\Gamma_{120} - ia\Gamma_{123} + b'(\Gamma_{453}+\Gamma_{783}+\Gamma_{693}) - ib(\Gamma_{450}+\Gamma_{690}+\Gamma_{780})\big) \\ &- \big(ie(\Gamma_{01245}+\Gamma_{01269}+\Gamma_{01278}) + f(\Gamma_{31245}+\Gamma_{31269}+\Gamma_{31278})\big)\Big) \\ &+ \text{term in (3.11)} \end{aligned} \tag{3.17}$$

Use of the explicit matrix representation for the Γ matrices simplifies eq (3.17) to

$$\begin{aligned} P_+\mathcal{N}_0P_-\kappa_1 = &\frac{1}{2}(1+\kappa_3)\Big(s + p\rho_3\eta_3 + a\rho_3 + a'\eta_3 \\ &+ (b' + b\rho_3\eta_3 + e\eta_3 + f\rho_3)(\sigma_3 + \tau_3 + \rho_3\tau_3\sigma_3) \\ &+ M_2(\frac{1}{2^5})(\rho_1 + i\rho_2)(\eta_1 + i\eta_2)(\tau_1 + i\tau_2)(\sigma_1 + i\sigma_2)\Big), \end{aligned} \tag{3.18}$$

where all terms containing η_3 break $SU(2)_L \times U(1)_Y$. Having specified all the vevs that break $SO(10)$ down to the low-energy symmetry, it is straightforward, though tedious, to write down the fermionic masses generated through the symmetry breaking. These are

$$\begin{aligned} I_{\text{fmass}} = -\frac{1}{\sqrt{3}}\int d^4x\Big(&\big[(s+p+3(e+f))K_{(pq)} + (a+a'+3(b+b'))K_{[pq]}\big]\overline{N_R^p}N_L^q \\ &+ \big[(s+p-(e+f))K_{(pq)} + (a+a'-(b+b'))K_{[pq]}\big]\overline{u_R^p}u_L^q \\ &+ \big[(s-p-3(e-f))K_{(pq)} + (a-a'-3(b-b'))K_{[pq]}\big]\overline{e_R^p}e_L^q \\ &+ \big((s-p+e-f)K_{(pq)} + (a-a'+b-b')K_{[pq]}\big)\overline{d_R^p}d_L^q \\ &+ \big[M_2K_{(pq)}(N_R^{pc})^TC^{-1}N_R^{qc}\big] + h.c\Big) \end{aligned} \tag{3.19}$$

where we have denoted the family mixing matrix K_{13} by K. For the neutral fields N_L and N_R we have a see-saw mechanism giving the right-handed neutrino a large Majorana mass [11-12], and the neutrino mass matrix takes the simple form (ignoring generation mixing)

$$\begin{array}{cc} & \begin{array}{cc} N_L & N_R^c \end{array} \\ \begin{array}{c} N_L \\ N_R^c \end{array} & \begin{pmatrix} 0 & m \\ m & M_2 \end{pmatrix} \end{array} \tag{3.20}$$

where m is of order of the weak scale. This matrix has two eigenstates of masses M_2 and $\frac{m^2}{M_2}$. The free parameters at this stage are $M_G,\ M_1,\ M_2,\ a,\ a',\ b,\ b',\ e,\ f,\ s$

and p and the matrix K_{pq}. However, when we will examine the scalar potential in the next section, it will become clear that, in order for the potential, or some terms in it, not to vanish the above parameters must be related. Also we note that, since both the symmetric and antisymmetric parts of K_{pq} enter the fermionic mass matrix, it cannot be completely removed. By performing a unitary transformation on $\chi_+^p \to U_q^p \chi_+^q$ such that $U^* U = 1$ the matrix K_{pq} is transformed to $(U^T K U)_{pq}$. Since K is an arbitrary complex matrix, the matrix U can be used to eliminate nine out of the eighteen real parameters. We shall come back to the fermionic mass terms after having examined the bosonic sector.

4. The bosonic action

In the non-commutative formulation of the Yang-Mills action, an essential ingredient is the Dirac operator. The curvature of the one-form ρ is defined by

$$\theta = d\rho + \rho^2. \tag{4.1}$$

The Yang-Mills action in the non-commutative setting is given by

$$I_b = \frac{1}{4} \mathrm{Tr}_\omega (\theta^2 |D|^{-4}) \tag{4.2}$$

where Tr_ω is the Dixmier trace. It was shown in [13] that one can equivalently use the heat-kernel expression

$$\lim_{\epsilon \to 0} \frac{\mathrm{tr}(\theta^2 e^{-\epsilon |D|^2})}{\mathrm{tr}(e^{-\epsilon |D|^2})} \tag{4.3}$$

For both definitions, it can be shown that the Yang-Mills action is equal to [4-5]

$$I = \frac{1}{4} \int d^4x \mathrm{Tr}\Big(\mathrm{tr}\big(\pi^2(\theta)\big)\Big) \tag{4.4}$$

To compute $\pi(\theta)$, the expression $\pi(d\rho)$ must be evaluated from the definition of ρ:

$$\pi(d\rho) = \sum_i [D, a^i][D, b^i] \tag{4.5}$$

and this must be expresssed in terms of the fields appearing in $\pi(\rho)$. Since $\pi(d\rho)$ is not necessarily zero when $\pi(\rho)$ is, one must quotient out the space $\mathrm{Ker}(\pi) + d\mathrm{Ker}(\pi)$. Since the Yang-Mills action is quadratic in the curvature θ, the process of working on the quotient space is equivalent to introducing non-dynamical auxiliary fields and eliminating them through there equations of motion. The Yang-Mills action in eq.(4.4) has been derived for an N point space in [10]. Here we simply quote the result:

$$I = \sum_{m=1}^{N} Tr\Big(\frac{1}{2}F_{\mu\nu}^{m}F^{\mu\nu m} - \Big|\sum_{p\neq m}|K_{mp}|^2|\phi_{mp}+M_{mp}|^2 - (Y_m + X'_{mm})\Big|^2$$
$$+ \sum_{p\neq m}|K_{mp}|^2\Big|\partial_\mu(\phi_{mp}+M_{mp}) + A_{\mu m}(\phi_{mp}+M_{mp}) - (\phi_{mp}+M_{mp})A_{\mu p}\Big|^2$$
$$- \sum_{n\neq m}\sum_{p\neq m,n}\Big|K_{mp}K_{pn}\big((\phi_{mp}+M_{mp})(\phi_{pn}+M_{pn}) - M_{mp}M_{pn}\big) - X_{mn}\Big)\Big|^2\Big) \tag{4.6}$$

where the A^m are the gauge fields in the $m-m$ entry of $\pi(\rho)$ and ϕ_{mn} are the scalar fields in the $m-n$ entry of $\pi(\rho)$. The X_{mn}, X'_{mn} and Y_m are fields whose unconstrained elements are auxiliary fields that can be eliminated from the action. Their expressions in terms of the a^i and b^i are

$$X_{mn} = \sum_i a_m^i \sum_{p\neq m,n} K_{mp}K_{pn}(M_{mp}M_{pn}b_n^i - b_m^i M_{mp}M_{pn}), \qquad m\neq n \tag{4.7}$$

$$X'_{mm} = \sum_i a_m^i \partial\!\!\!/^2 b_m^i + (\partial^\mu A_\mu^m + A^{\mu m}A_\mu^m) \tag{4.8}$$

$$Y_m = \sum_{p\neq m}\sum_i a_m^i|K_{mp}|^2|M_{mp}|^2 b_m^i \tag{4.9}$$

In the case at hand the discrete space has three points. Because of the permutation and complex conjugation symmetry, the a_m^i are related to each other. This in turn relates some of the auxiliary fields to one another. To use eq.(4.5), we must compute the different terms as functionals of the component fields appearing in $\pi(\rho)$. We first write

$$A = \frac{g}{4}\gamma^\mu A_\mu^{IJ}\Gamma_{IJ}$$

where g is the $SO(10)$ gauge coupling constant. Then the kinetic term for the gauge field A_μ^{IJ} as given by the first term in eq (4.6), after computing the sum and the trace over $\mathrm{Cliff}\big(SO(10)\big)$, is equal to

$$-4g^2F_{\mu\nu}^{IJ}F^{\mu\nu IJ} \tag{4.10}$$

where the field strength is

$$F_{\mu\nu}^{IJ} = \partial_\mu A_\nu^{IJ} - \partial_\nu A_\mu^{IJ} + g(A_\mu^{IK}A_\nu^{KJ} - A_\nu^{IK}A_\mu^{KJ}) \tag{4.11}$$

The Higgs kinetic terms have two parts, corresponding to $\mathcal{M}$ and $\mathcal{N}$. Using the decompositon of $\mathcal{M}$ and $\mathcal{N}$ in the $\mathrm{Cliff}\big(SO(10)\big)$-basis one gets the result

$$\begin{aligned}&64\mathrm{Tr}|K_{12}|^2\Big(\big((\partial_\mu m)^2\big) + 2\big(D_\mu(m+m_0)_{IJ}\big)^2 + 4\big(D_\mu(m+m_0)_{IJKL}\big)^2\Big)\\ &+ 64|K_{13}|^2\Big(|D_\mu(n+n_0)_I|^2 + 3|D_\mu(n+n_0)_{IJK}|^2 + 5|D_\mu(n+n_0)_{IJKLM}|^2\Big)\end{aligned} \tag{4.12}$$

where the D appearing in this equation is the covariant derivative with respect to the $SO(10)$ gauge group, and the m's and n's are defined in eq. (2.19). For example $D_\mu n_I = \partial_\mu n_I + g A_\mu^{IJ} n_J$. The masses of the components of the gauge fields A_μ^{IJ} corresponding to the broken generators of $SO(10)$ are provided by the vevs $\mathcal{M}_0$, and $\mathcal{N}_0$. The most complicated part is the Higgs potential, since this involves new fields some of which are related, and the non-dynamical ones must be eliminated through their equations of motion. It is given by

$$
\begin{aligned}
V(\mathcal{M},\mathcal{N}) = 2\Big| &|K_{12}|^2|\mathcal{M}+\mathcal{M}_0|^2 + |K_{13}|^2|\mathcal{N}+\mathcal{N}_0|^2 - (Y_1 + X'_{11})\Big|^2 \\
&+ \Big||K_{31}|^2|\mathcal{N}+\mathcal{N}_0|^2 + |K_{12}|^2|\mathcal{M}+\mathcal{M}_0|^2 - (Y_3 + X'_{33})\Big|^2 \\
&+ 2\Big||K_{13}|^2|(\mathcal{N}+\mathcal{N}_0|^2 - |\mathcal{N}_0|^2)|^2 - X_{12}\Big|^2 \\
&+ 2\Big|K_{12}K_{23}\big((\mathcal{M}+\mathcal{M}_0)(\mathcal{N}+\mathcal{N}_0) - \mathcal{M}_0\mathcal{N}_0\big) - X_{13}\Big|^2,
\end{aligned}
\tag{4.13}
$$

where we have used the symmetry that equates some of the $K's$ and of the $X's$. We now write the explicit expressions for the X and Y fields. First, we have :

$$
\begin{aligned}
X'_{11} &= \sum_i a^i \partial^2 b^i + (\partial^\mu A_\mu + A^\mu A_\mu) \\
X'_{33} &= B\overline{X'_{11}}B^{-1}
\end{aligned}
\tag{4.14}
$$

Next, we have for the Y's

$$
\begin{aligned}
Y_1 &= \sum_i a^i |K_{12}|^2 |\mathcal{M}_0|^2 b^i + 2\sum_i a^i |K_{13}|^2 |\mathcal{N}_0|^2 b^i \\
Y_3 &= B\overline{Y_1}B^{-1}
\end{aligned}
\tag{4.15}
$$

Finally we have for the $X_{mn}, m \neq n$, the expressions

$$
\begin{aligned}
X_{12} &= |K_{13}|^2\Big(\sum_i a^i |\mathcal{N}_0|^2 B\overline{b^i}B^{-1} - |\mathcal{N}_0|^2\Big) \\
X_{13} &= K_{12}K_{23}\Big(\sum_i a^i \mathcal{M}_0\mathcal{N}_0 B\overline{b^i}B^{-1} - \mathcal{M}_0\mathcal{N}_0\Big),
\end{aligned}
\tag{4.16}
$$

and the other X's are related to the above ones by permutation symmetry. It is easy to notice that X'_{11} and X'_{33} are auxiliary fields that do not depend on the K matrices. Therefore, eliminating these fields would result in expressions orthogonal to the corresponding K space. Eliminating the remaining auxiliary fields $Y_1,\ Y_3,\ X_{12}$, and X_{13} is much more complicated. If all of these were independent the potential would vanish, after eliminating them. However, if the vevs $\mathcal{M}_0$ and $\mathcal{N}_0$ are chosen in a special way then it is possible for the potential to survive. One must arrange for a relation between the auxiliary fields, so that, after eliminating the independent combinations, the potential that corresponds to the given vacuum will result. A close look at the potential in eq. (4.13) shows that if all of the X and Y fields are independent, the potential disappears after eliminating them. By comparing X_{12} and Y_1 one sees that they can be related only if $\sum a^i |\mathcal{M}_0|^2 b^i$ is not an independent field. This can happen if

$$M_G = M_1 \tag{4.17}$$

so that $|\mathcal{M}_0|^2 = 4M_1^2$, and we get the relation

$$Y_1 = |K_{12}|^2 |\mathcal{M}_0|^2 + |K_{13}|^2 |\mathcal{N}_0|^2 + X_{12} \tag{4.18}$$

Next, for the term in the potential depending on X_{13} not to vanish, X_{13} must not be an independent field and must be a function of $\mathcal{N}$. This is possible if $\mathcal{M}_0 \mathcal{N}_0$ is proportional to $\mathcal{N}_0$. This condition is extremely restrictive, but fortunately has one solution given by

$$\begin{aligned} \mathcal{M}_0 \mathcal{N}_0 &= 2M_1 \mathcal{N}_0 \\ a' &= b' = 0 \\ f &= -s = \frac{a}{2} \\ p &= 3e = \frac{3}{2} b \end{aligned} \tag{4.19}$$

and the free parameters in the theory are $M_1,\ M_2,\ a,\ b$ and the matrices $K_{12},\ K_{13}$. The equation for X_{13} simplifies to

$$X_{13} = K_{13}(2M_1 \mathcal{N}) \tag{4.20}$$

Then the only independent fields to be eliminated are X_{12} and $X_{11}^{'}$. The resulting potential is

$$\begin{aligned} V(\mathcal{M}, \mathcal{N}) = &\left(\mathrm{Tr}|K_{12}|^4 - (\mathrm{Tr}|K_{12}|^2)^2\right) \Big| |\mathcal{M} + \mathcal{M}_0|^2 - 4M_1^2 \Big|^2 \\ &+ 2\mathrm{Tr}|K_{12}K_{13}|^2 \Big| (\mathcal{M} + \mathcal{M}_0 - 2M_1)(\mathcal{N} + \mathcal{N}_0) \Big|^2 \end{aligned} \tag{4.21}$$

The total bosonic action is the sum of the terms (4.10), (4.12) and (4.21), multiplied by an overall constant. We choose this constant to be $\frac{1}{16g^2}$ to get the canonical kinetic energy for the gauge fields. The kinetic energy for the scalar fields, $\mathcal{M}$ and $\mathcal{N}$, is normalized canonically after rescaling

$$\begin{aligned} \mathcal{M} &\to \frac{g}{2\sqrt{2\mathrm{Tr}|K'|^2}} \mathcal{M} \\ \mathcal{N} &\to \frac{g}{2\sqrt{\mathrm{Tr}|K|^2}} \mathcal{N}, \end{aligned} \tag{4.22}$$

where we have denoted K_{13} by K and K_{12} by K'. After rescaling, the bosonic action becomes

$$\begin{aligned} I_{\text{bosonic}} = \int d^4x \Big(& -\frac{1}{4} F^{IJ}_{\mu\nu} F^{\mu\nu IJ} \\ & + \frac{1}{32} \mathrm{Tr} \Big[\frac{1}{2} \big(D_\mu (\mathcal{M} + \mathcal{M}_0) \big)^2 + \big| D_\mu (\mathcal{N} + \mathcal{N}_0) \big|^2 \Big] \\ & + \frac{g^2}{2^5.32} \Big(\frac{\mathrm{Tr}|K'|^4}{(\mathrm{Tr}|K'|^2)^2} - 1 \Big) \mathrm{Tr} \Big| |\mathcal{M} + \mathcal{M}_0|^2 - 4M_1^2 \Big|^2 \\ & + \frac{g^2}{2^3.32} \Big| (\mathcal{M} + \mathcal{M}_0 - 2M_1)(\mathcal{N} + \mathcal{N}_0) \Big|^2 \Big) \end{aligned} \tag{4.23}$$

Finally, the fermionic action becomes

$$\begin{aligned} I_{\mathrm{f}} = -\frac{g}{\sqrt{3\mathrm{Tr}|K|^2}} \int d^4x \Big(& K_{pq} \big((a+3b) \overline{N^p_R} N^q_L + (a-3b) \overline{e^p_R} e^q_L \big) \\ & + K_{qp} \big((-a+b) \overline{u^p_R} u^q_L - (a+b) \overline{d^p_R} d^q_L \big) + M_2 K_{(pq)} (N^p_R)^T C^{-1} N^q_R + h.c \Big) \end{aligned} \tag{4.24}$$

By examining the gauge kinetic term one finds the usual $SO(10)$ relations among the gauge coupling constants

$$g_2 = g_3 = g = \sqrt{\frac{5}{3}} g_1 \tag{4.25}$$

implying that $\sin^2\theta_W$, at the unification scale M_1 is $\frac{3}{8}$. From the $\mathcal{N}$-kinetic term one sees that the mass of the W gauge boson is

$$m_W^2 = \frac{g^2}{4}(a^2 + 3b^2) \tag{4.26}$$

From the fermionic mass terms, one deduces, using the fact that the top quark mass is much heavier than the other fermionic masses, that

$$m_t = g|b - a| \tag{4.27}$$

Comparing with m_W we get the relation

$$m_t = 2m_W \frac{|1 - \frac{b}{a}|}{\sqrt{1 + \frac{3b^2}{a^2}}} \tag{4.28}$$

and this gives upper and lower bounds on the top quark mass

$$\frac{2}{\sqrt{3}} m_w \leq m_t \leq \frac{4}{\sqrt{3}} m_w = 186.13\mathrm{Gev} \tag{4.29}$$

which agrees with present experimental limits. Unfortunately, the same matrix K_{qp} appears for the u^p and d^p quarks, implying that the same transformation can be used for u^p and d^p to diagonalize K_{qp}. This in turn implies that this model does not allow for a Cabibbo angle, and this phenomenologically rules out this model. This forces us to look for modifications in this model so that it becomes acceptable. This result shows that model building in non-commutative geometry is so constrained that the models could be ruled out on phenomenological grounds.

5. A realistic SO(10) model

The model presented in the previous sections is minimal in the sense that the number of points in the internal geometry and the Higgs fields cannot be reduced. If one insists on a Z_2 symmetry between the different copies, then the number of points would have to be even, and we have to take two copies where the conjugate spinors are placed, instead of the one copy considered before. It will be seen that this extension cannot have a potential after eliminating the auxiliary fields. Therefore, this model has to be further extended by one or two points to get the $\underline{16}_s$ Higgs field, and this will ensure that the potential can be arranged to survive. The fermionic space is extended with a singlet spinor. Two of the neutral fermions will become superheavy, while the third one would remain massless. The triple $(\mathcal{A}, h, D)$ is defined in the same way as in section 2, with the algebra $\mathcal{A}_2$ given by

$$\mathcal{A}_2 \equiv P_+ \text{Cliff}\big(SO(10)\big) P_+ \oplus R, \tag{5.1}$$

The involutive map π is now taken to be:

$$\pi(a) = \pi_0(a) \oplus \pi_0(a) \oplus \overline{\pi}_0(a) \oplus \overline{\pi}_0(a) \oplus \pi_1(a) \oplus \pi_1(a) \tag{5.2}$$

acting on the Hilbert space

$$\tilde{h} = h_1 \otimes (h_2^{(1)} \oplus \cdots \oplus h_2^{(6)}) \tag{5.3}$$

where $h_2^{(i)} \cong h_2, \qquad i = 1 \cdots 4$, and $h_2^{(i)} \cong C \qquad i = 5, 6$. Let h denote the subspace of $\tilde{h}$ which is the image of the orthogonal projection into elements of the form

$$\Psi \equiv \begin{pmatrix} P_+\psi \\ P_+\psi \\ P_-\psi^c \\ P_-\psi^c \\ \lambda \\ \lambda^c \end{pmatrix}, \tag{5.4}$$

On $\tilde{h}$ the self-adjoint Dirac operator D becomes

$$D = \begin{pmatrix} \partial\!\!\!/ \otimes 1 \otimes 1 & \gamma_5 \otimes M_{12} \otimes K_{12} & \ldots & \gamma_5 \otimes M_{16} \otimes K_{16} \\ \gamma_5 \otimes M_{21} \otimes K_{21} & \partial\!\!\!/ \otimes 1 \otimes 1 & \ldots & \gamma_5 \otimes M_{26} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_5 \otimes M_{61} \otimes K_{61} & \gamma_5 \otimes M_{62} \otimes K_{62} & \ldots & \partial\!\!\!/ \otimes 1 \end{pmatrix} \tag{5.5}$$

where the K_{mn} are 3×3 matrices commuting with the a_i and b_i. Therefore we shall take

$$
\begin{aligned}
&M_{12}=M_{21}=\mathcal{M}_0\\
&M_{34}=M_{43}=BM_{12}B^{-1}\\
&M_{13}=M_{23}=M_{14}=M_{24}=\mathcal{N}_0\\
&M_{15}=M_{16}=M_{25}=M_{26}=H_0\\
&M_{35}=M_{45}=M_{36}=M_{46}=BM_{15}\\
&M_{56}=0
\end{aligned}
\tag{5.6}
$$

where $\mathcal{M}_0$ and $\mathcal{N}_0$ are given by eq. (2.12). Similar symmetry conditions are imposed on K_{mn}. For $\pi(\rho)$ one then gets

$$
\pi(\rho)=\begin{pmatrix}
A & \gamma_5\mathcal{M}K_{12} & \gamma_5\mathcal{N}K_{13} & \gamma_5\mathcal{N}K_{14} & \gamma_5HK_{15} & \gamma_5HK_{16}\\
\gamma_5\mathcal{M}K_{12} & A & \gamma_5\mathcal{N}K_{23} & \gamma_5\mathcal{N}K_{24} & \gamma_5HK_{25} & \gamma_5HK_{26}\\
\gamma_5\mathcal{N}^*K_{31} & \gamma_5\mathcal{N}^*K_{32} & B\overline{A}B^{-1} & \gamma_5\mathcal{M}'K_{34} & \gamma_5H'K_{35} & \gamma_5H'K_{36}\\
\gamma_5\mathcal{N}^*K_{41} & \gamma_5\mathcal{N}^*K_{42} & \gamma_5\mathcal{M}^{'}K_{43} & B\overline{A}B^{-1} & \gamma_5H'K_{45} & \gamma_5H'K_{46}\\
\gamma_5H^*K_{51} & \gamma_5H^*K_{52} & \gamma_5H^{'*}K_{53} & \gamma_5H^{'*}K_{54} & 0 & 0\\
\gamma_5H^*K_{61} & \gamma_5H^*K_{62} & \gamma_5H^{'*}K_{63} & \gamma_5H^{'*}K_{64} & 0 & 0
\end{pmatrix}
\tag{5.7}
$$

where the new functions A, $\mathcal{M}$, $\mathcal{N}$ and H are given in terms of the a^i and b^i by

$$
\begin{aligned}
A&=P_+(\sum_i a^i\partial\!\!\!/ b^i)P_+\\
\mathcal{M}+\mathcal{M}_0&=P_+(\sum_i a^i\mathcal{M}_0b^i)P_+\\
\mathcal{N}+\mathcal{N}_0&=P_+(\sum_i a^i\mathcal{N}_0B\overline{b^i}B^{-1})P_-\\
H+H_0&=P_+(\sum_i a^iH_0b^{'i})
\end{aligned}
\tag{5.8}
$$

and

$$
\mathcal{M}^{'}=B\overline{\mathcal{M}}B^{-1},\tag{5.9}
$$

$$
H'=B\overline{H}.\tag{5.10}
$$

We shall make the same physical requirement as in eqs. (2.20) and (2.21) that reduces the gauge group from $U(16)$ to $SO(10)$. The fermionic action, in terms of the component fields, is given by

$$
\begin{aligned}
I_f=\int d^4x\Big(&2\,\overline{\psi_+}\Big[i(\partial\!\!\!/+A)\psi_++2\gamma_5(\mathcal{N}+\mathcal{N}_0)\psi_+^cK_{13}+\gamma_5(H+H_0)\lambda^cK_{15}\Big]\\
&-2\,\overline{\psi_+^c}\Big[i(\partial\!\!\!/+A)\psi_+^c+2\gamma_5(\mathcal{N}^*+\mathcal{N}_0^*)\psi_+K_{13}^*\gamma_5B(\overline{H}+\overline{H_0})\lambda K_{15}^*\Big]\\
&+\overline{\lambda}\Big[i\partial\!\!\!/\lambda+2\gamma_5(H+H_0)^TB^{-1}\psi_+^cK_{15}+\Big]-\overline{\lambda^c}\Big[i\partial\!\!\!/\lambda^c+2\gamma_5(H^*+H_0^*)\psi_+K_{15}^*\Big]\Big)
\end{aligned}
\tag{5.11}
$$

where $\psi_+ = P_+\psi$ and by $SO(10)$ chirality is equal to ψ. This expression can be simplified by using the proporties of the charge conjugation matrices B and C and, after rescaling $\psi \to \frac{1}{2}\psi$ and $\lambda \to \frac{1}{\sqrt{2}}\lambda$, the fermionic action (5.11) simplifies to

$$\begin{aligned}I_f = \int d^4x\Big(&\overline{\psi_+}i(\partial\!\!\!/ + A)\psi_+ + \overline{\lambda}i\partial\!\!\!/\lambda \\ &- \Big[\psi_+^T B^{-1}C^{-1}(\mathcal{N}^* + \mathcal{N}_0^*)\psi_+ K_{13}^* \\ &+ \frac{1}{\sqrt{2}}\lambda^T C^{-1}(H^* + H_0^*)\psi_+ K_{15}^* + \frac{1}{\sqrt{2}}\psi_+^T C^{-1}(\overline{H} + \overline{H_0})\lambda K_{35} + h.c\Big]\Big)\end{aligned} \tag{5.12}$$

The only change in the breaking mechanism is that $U(1)_c \times SU(2)_R$ is broken also by the H_0 whose vev is given by

$$H_0 = M_3 \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \tag{5.13}$$

The fermionic action is modified slightly from eq (3.19) to become

$$\begin{aligned}I_{\text{f}-\text{mass}} = -\int d^4x\Big(&\big((s+p+3(e+f))K_{(pq)} + (a+a'+3(b+b'))K_{[pq]}\big)\overline{N_R^p}N_L^q \\ &+ \big((s+p-(e+f))K_{(pq)} + (a+a'-(b+b'))K_{[pq]}\big)\overline{u_R^p}u_L^q \\ &+ \big((s-p-3(e-f))K_{(pq)} + (a-a'-3(b-b'))K_{[pq]}\big)\overline{e_R^p}e_L^q \\ &+ \big((s-p+e-f)K_{(pq)} + (a-a'+b-b')K_{[pq]}\big)\overline{d_R^p}d_L^q \\ &+ \big(\sqrt{2}M_3 K'_{pq}\overline{N_R^p}\lambda_L^q + M_2 K_{(pq)}(N_R^{pc})^T C^{-1}N_R^{qc}\big) + h.c\Big)\end{aligned} \tag{5.14}$$

where we have denoted the family mixing matrices K_{13}, K_{15} and K_{56} by K, K', K'', respectively. Since we have three neutral fields, N_L, N_R^c and λ_L, and their mass eigenstates are mixed, the mass matrix must be diagonalised. Ignoring the mixing due to the generation matrices, the mass matix is of the form

$$\begin{array}{c c} & \begin{array}{ccc} N_L & N_R^c & \lambda_L \end{array} \\ \begin{array}{c} N_L \\ N_R^c \\ \lambda_L \end{array} & \begin{pmatrix} 0 & m & 0 \\ m & M_2 & M_3 \\ 0 & M_3 & 0 \end{pmatrix} \end{array} \tag{5.15}$$

and we shall assume a mass hierarchy $m \ll M_2, M_3$, and $M_2 \sim M_3$. Diagonalisation of the matrix (5.13) produces two massive fields whose masses are of order M_2, and the third will be a massless left-handed neutrino. The kinetic term for the gauge field A_μ^{IJ} is equal to

$$-4g^2 F_{\mu\nu}^{IJ}F^{\mu\nu IJ} \tag{5.16}$$

and the Higgs kinetic terms have three parts corresponding to $\mathcal{M}$, $\mathcal{N}$ and H. They are given by

$$\begin{aligned}&2|K_{12}|^2\mathrm{Tr}\Big(\big(D_\mu(\mathcal{M}+\mathcal{M}_0)\big)^2\Big)+8|K_{13}|^2\mathrm{Tr}\Big(|D_\mu(\mathcal{N}+\mathcal{N}_0)|^2\Big)\\&+12|K_{15}|^2\Big|D_\mu(H+H_0)\Big|^2\end{aligned}\tag{5.17}$$

where the D appearing in this equation is the covariant derivative with respect to the $SO(10)$ gauge group. The mass terms of the gauge fields corresponding to the broken generators of $SO(10)$ are provided by the vevs $\mathcal{M}_0$, $\mathcal{N}_0$ and H_0. The Higgs potential is very complicated in this case. It is given by

$$\begin{aligned}&2\Big||K_{12}|^2|\mathcal{M}+\mathcal{M}_0|^2+2|K_{13}|^2|\mathcal{N}+\mathcal{N}_0|^2+2|K_{15}|^2|H+H_0|^2-(Y_1+X'_{11})\Big|^2\\&+2\Big|2|K_{31}|^2|\mathcal{N}+\mathcal{N}_0|^2+|K_{12}|^2|\mathcal{M}+\mathcal{M}_0|^2+2|K_{13}|^2|H+H_0|^2-(Y_3+X'_{33})\Big|^2\\&+2\Big|4|K_{51}|^2H+H_0|^2-(Y_5+X'_{55})\Big|^2\\&+2\Big|2|K_{13}|^2(|\mathcal{N}+\mathcal{N}_0|^2-|\mathcal{N}_0|^2)+2|K_{15}|^2(|H+H_0|^2-|H_0|^2)-X_{12}\Big|^2\\&+8\Big|K_{12}K_{23}\big((\mathcal{M}+\mathcal{M}_0)(\mathcal{N}+\mathcal{N}_0)-\mathcal{M}_0\mathcal{N}_0\big)\\&\qquad+K_{14}K_{43}\big((\mathcal{N}+\mathcal{N}_0)(\overline{\mathcal{M}}+\overline{\mathcal{M}_0})-\mathcal{N}_0\overline{\mathcal{M}_0}\big)\\&\qquad+2K_{15}K_{53}\big((H+H_0)B(\overline{H}+\overline{H_0}-H_0B\overline{H_0}\big)-X_{13}\Big|^2\\&+8\Big|K_{12}K_{25}(\mathcal{M}+\mathcal{M}_0)(H+H_0)+2K_{13}K_{35}(\mathcal{N}+\mathcal{N}_0)B(\overline{H}+\overline{H_0})-X_{15}\Big|^2\\&+2\Big|2K_{31}K_{14}\big(|\mathcal{N}^*+\mathcal{N}_0^*|^2-|\mathcal{N}_0^*|^2\big)+2|K_{35}|^2\big(|B(\overline{H}+\overline{H_0})|^2-|B\overline{H_0}|^2\big)-X_{34}\Big|^2\\&+8\Big|2K_{31}K_{15}\big((\mathcal{N}^*+\mathcal{N}_0^*)(H+H_0)-\mathcal{N}_0^*H_0^*\big)\\&\qquad+2K_{34}K_{45}\big(|B(\overline{H}+\overline{H_0})|^2-|B\overline{H_0}|^2\big)-X_{35}\Big|^2\\&+2\Big|4|K_{51}|^2\big(|H^*+H_0^*|^2-|H_0^*|^2\big)-X_{56}\Big|^2\end{aligned}\tag{5.18}$$

- where we have used the symmetry that equates some of the $K's$ and the $X's$. The explicit expressions for the X and Y fields are:

$$\begin{aligned}X'_{11}&=\sum_i a^i\partial^2 b^i+(\partial^\mu A_\mu+A^\mu A_\mu)\\X'_{33}&=B\overline{X'_{11}}B^{-1}\\X'_{55}&=\sum_i a'^i\partial^2 b'^i\end{aligned}\tag{5.19}$$

Next, we have

$$\begin{aligned}
Y_1 &= \sum_i a^i |K_{12}|^2 |\mathcal{M}_0|^2 b^i + 2\sum_i a^i |K_{13}|^2 |\mathcal{N}_0|^2 b^i + \sum_i a^i |K_{15}|^2 |H_0|^2 b^i \\
Y_3 &= B\overline{Y_1}B^{-1} \\
Y_5 &= 2M_2^2(|K_{51}|^2 + |K_{53}|^2)
\end{aligned} \tag{5.20}$$

The expressions for $X_{mn}, m \neq n$ are now given by

$$\begin{aligned}
X_{12} &= 2|K_{13}|^2 \big(\sum_i a^i |\mathcal{N}_0|^2 B\overline{b^i}B^{-1} - |\mathcal{N}_0|^2\big) + 2|K_{15}|^2 \big(\sum_i a^i |H_0|^2 b^i - |H_0|^2\big) \\
X_{13} &= K_{12}K_{23}\big(\sum_i a^i \mathcal{M}_0\mathcal{N}_0 B\overline{b^i}B^{-1} - \mathcal{M}_0\mathcal{N}_0\big) \\
&\qquad + K_{14}K_{43}\sum_i a^i \mathcal{N}_0 B\overline{\mathcal{M}_0}B^{-1} - \mathcal{N}_0 B\overline{\mathcal{M}_0}B^{-1}\big) \\
&\qquad + 2|K_{15}|^2\big(\sum_i a^i H_0\overline{H_0}\overline{b^i}B^{-1} - |H_0|^2 B^{-1}\big) \\
X_{15} &= |K_{12}K_{25}|\big(\sum_i a^i \mathcal{M}_0 H_0 b^{'i} - \mathcal{M}_0 H_0\big) + 2K_{13}K_{35}\big(\sum_i a^i \mathcal{N}_0 B\overline{H_0}b^{'i} - \mathcal{N}_0 B\overline{H_0}\big) \\
X_{34} &= B\overline{X_{12}}B^{-1} \\
X_{35} &= B\overline{X_{15}} \\
X_{56} &= 0
\end{aligned} \tag{5.21}$$

and the other X's are related to the ones above by permutation symmetry $X_{12} = X_{21}$, $X_{34} = X_{43}$, $X_{13} = X_{14} = X_{23} = X_{24}$, $X_{16} = X_{26}$ and $X_{36} = X_{46}$. We also have similar identities for the K's, and, in addition, we have assumed the relations $K_{12} = \overline{K_{34}}$ and $K_{15} = \overline{K_{35}}$. In analogy with the previous model, we must impose the relation

$$M_G = M_1, \tag{5.22}$$

in order to get a relation between X_{12} and Y_1:

$$Y_1 = |K_{12}|^2|\mathcal{M}_0|^2 + 2|K_{13}|^2|\mathcal{N}_0|^2 + X_{12} \tag{5.23}$$

For the term in the potential involving X_{13} to survive, we should be able to express this field in terms of the other scalar fields. By examining the expression for X_{13} we notice that a simplification occurs if we require that

$$K_{12} = \overline{K_{12}} \tag{5.24}$$

because the terms involving $\mathcal{M}_0\mathcal{N}_0^{(1)}$ drop out, where $\mathcal{N}_0^{(1)}$ is the part of $\mathcal{N}_0$ independent of M_2. In this case X_{13} can be made to be zero, provided that we take

$$M_1 M_2 = -\frac{K_{15}\overline{K_{15}}}{2K_{12}K_{13}} M_3^2 \tag{5.25}$$

where we have used the relation

$$H_0 H_0^* = \frac{M_2}{M_3^2} \mathcal{N}_0^{(2)} \tag{5.26}$$

where $\mathcal{N}_0^{(2)}$ is the part in $\mathcal{N}_0$ dependent on M_2. If no relation is taken between K_{12} and $\overline{K_{12}}$ then the only way for the potential to survive is to impose a relation on $\mathcal{M}_0\mathcal{N}_0$ identical to the one for the simpler model as well as a relation between M_2 and M_3. This case will not be interesting for us since the fermionic mass matrices, apart from the neutral fields sector, are identical to those in the previous model and thus would suffer the same problem of the absence of the Cabibbo angle. The auxiliary field X_{15} is not independent and is equal to

$$X_{15} = uH, \tag{5.27}$$

where

$$u = 2K_{13}\overline{K_{15}}\Big(s + p - 3(b + b') + 2(a + a') + M_2\Big) - 2K_{12}K_{25}M_1 \tag{5.28}$$

After eliminating the auxiliary fields Y_1 and $X_{33}^{'}$ the potential becomes

$$\begin{aligned}
V(\mathcal{M}, \mathcal{N}, H) = & \big(\mathrm{Tr}|K_{12}|^4 - (\mathrm{Tr}|K_{12}|^2)^2\big)\mathrm{Tr}\Big(|\mathcal{M} + \mathcal{M}_0|^2 - |\mathcal{M}_0|^2\Big)^2 \\
& + 4\Big|K_{13}K_{12}\big((\mathcal{M} + \mathcal{M}_0)(\mathcal{N} + \mathcal{N}_0) + (\mathcal{N} + \mathcal{N}_0)B(\overline{\mathcal{M}} + \overline{\mathcal{M}_0})B^{-1}\big) \\
& \qquad + 2K_{15}\overline{K_{15}}\big((H + H_0)B(\overline{H} + \overline{H_0})\big)\Big|^2 \\
& + 8\Big|K_{12}K_{15}(\mathcal{M} + \mathcal{M}_0)(H + H_0) + 2K_{13}\overline{K_{15}}(\mathcal{N} + \mathcal{N}_0)B(\overline{H} + \overline{H_0}) \\
& \qquad - u(H + H_0)\Big|^2 \\
& + 16\big(\mathrm{Tr}|K_{15}|^4 - (\mathrm{Tr}|K_{15}|^2)^2\big|\big||H^* + H_0^*|^2 - M_3^2\big|^2 \\
& + 16\mathrm{Tr}|K_{15}|^4\Big||H^* + H_0^*|^2 - M_3^2\Big|^2
\end{aligned} \tag{5.29}$$

Therefore the fermionic mass terms are still given by eq.(5.14) and do not suffer the problem encountered before This completes our study of the model and shows that it is possible to obtain a nice $SO(10)$ model. A complete phenomenological analysis will be left for the future.

6.Summary and conclusion

We have seen that a realistic $SO(10)$ model can be constructed using the non-commutative geometry setting of Connes. The attractiveness of this model stems from the fact that all the fermions fit into one representation making the spinor space particularly simple. Depending on the number of discrete points extending the continuous geometry the Higgs structure is predicted uniquely. We found two models: The first one is quite simple and has a very restrictive form for the fermion masses, but turns out to be unrealistic. The second example is more complicated, but the Higgs structure is essentially the same as that of the first model, with the difference of an additional $\underline{16_s}$ Higgs field. The fermionic masses are not as restricted as those in the first model. We hope to study the spectrum in more detail, in the future. A study of the quantum system is not meaningful before having determined those symmetries of the system that are characteristic of the non-commutative geometry setting.

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